Advanced Mathematical Models & Applications
Vol.5, No.1, 2020, pp.131-139

AN ALTERNATING PROCEDURE WITH DYNAMIC RELAXATION FOR CAUCHY PROBLEMS GOVERNED BY THE MODIFIED HELMHOLTZ EQUATION

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Abstract. In this paper, two relaxation algorithms on the Dirichlet Neumann boundary condition, for solving the Cauchy problem governed to the Modified Helmholtz equation are presented and compared to the classical alternating iterative algorithm. The numerical results obtained using our relaxed algorithm and the finite element approximation show the numerical stability, consistency and convergence of these algorithms. This confirms the efficiency of the proposed methods.

Keywords: Inverse Cauchy problem, modified Helmholtz equation, relaxed alternating iterative method, numerical simulation.

AMS Subject Classification: 65N21(35J25, 35R30).

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Received: 15 November 2019; Revised: 14 March 2020; Accepted: 15 April 2020; Published: 29 April 2020.

1 Introduction

In General, boundary value problems for the Modified Helmholtz equation reads as follow

$$\Delta u - k^2 u = f,$$

where a real value of k is the wave number, is often encountered in many branches of science and engineering. This equation is used to model a wide variety of physical phenomena. These include among others, wave propagation, vibration phenomena, aeroacoustic.

We are interested in the Cauchy problem where the boundary conditions for both the solution and its normal derivative are prescribed only on a part of the boundary of the solution domain, whilst no information is available on the other part of the boundary (Bergam et al., 2019; Choulli, 2009; Isakov, 2017; Kabanikhin, 2012).

For the Cauchy problem for governed by the Poisson equation Jourhmane & Nachaoui (1996, 1999, 2002) proposed the relaxation of the given Dirichlet data in the case of the alternating iterative algorithm proposed in Kozlov et al. (1991) This procedure drastically reduced the number of iterations required to achieve convergence for the inverse problems considered. It was used in Essaouini et al. (2004) and Essaouini & Nachaoui. (2004) for a non linear elliptic problem, in elasticity (Ellabib & Nachaoui, 2008; Marin & Johansson, 2010), and recently for Cauchy problem governed by Stocks equation (Chakib et al., 2018). Other methods have been developed for solving Cauchy's problems. The reader can consult for example Berntsson et al. (2017); Chakib et al. (2006); Choulli (2009); Isakov (2017); Kabanikhin (2012);

Lavrent'ev (2013); Nachaoui (2003); Qian et al. (2010); Shi et al. (2009) and the references therein. Our objective here is to propose a relaxation by dynamic parameters to solve the Cauchy problem governed by Hemholtz equation. We how numerically that the application of this algorithm considerably reduces the number of iterations to reach convergence compared to the classical algorithm which suffers as it was shown in the literature of a slowness which sometimes makes it unusable.

The paper is organized as follow. In section 2, we presents the mathematical formulation, on the basis of dealing with modified Helmholtz equations. In section 3, we describe the different alternating algorithms. Finally, section 6 is devoted to the numerical results and discussions.

2 Description of algorithms

Let Ω be a bounded domain in $\mathbb{R}^d$ with the Lipschitz boundary Γ, where d is the space dimension in which the problem is posed, usually $d \in \{1, 2, 3\}$. Let Γ be divided into two disjoint parts Γ_0 and Γ_1. We denote ν the outward unite normal to the boundary Ω and consider the following Cauchy problem for the Modified Helmholtz equation:

$$\Delta u - k^2 u = 0, \quad \text{in} \quad \Omega, \tag{1}$$

subject to the the following boundary conditions

$$u = f_1, \quad \text{on} \quad \Gamma_1, \tag{2}$$
$$\partial_\nu u = f_2, \quad \text{on} \quad \Gamma_1, \tag{3}$$

where the wave number k is positive real constant, ∂_ν denotes the outward normal derivative of u, f_1 and f_2 are known Cauchy data on Γ_1. The Cauchy problem is an ill-posed problem in the sense of Hadamard (1953).

2.1 The classical alternating algorithms

The alternating procedure is an iterative algorithm for solving Cauchy problems for laplace equation was introduced by Kozlov-Maz'ya in Kozlov et al. (1991).

This alternating procedure for the problem (1)-(3) consists in solving alternatively two auxiliary problems defined, respectively by

$$\Delta u - k^2 u = 0, \quad \text{in } \Omega, \tag{4}$$
$$u = v, \quad \text{on } \Gamma_0, \tag{5}$$
$$\partial_\nu u = f_2, \quad \text{on } \Gamma_1, \tag{6}$$

and

$$\Delta u - k^2 u = 0, \quad \text{in } \Omega, \tag{7}$$
$$\partial_\nu u = \eta, \quad \text{on } \Gamma_0, \tag{8}$$
$$u = f_1, \quad \text{on } \Gamma_1, \tag{9}$$

where f_1 and f_2 are given in (1)-(3), while v and η are two functions will be changing in each iteration. Problems (4)-(6) and (7)-(9) should alternately be solved until a prescribed stopping criterion is satisfied. This algorithm can be summarized by

Algorithm 1: Classical approach as proposed in Kozlov et al. (1991)

1: $n \leftarrow 0$, choose the initial guess $v = v^0$, then
2: Find $u^{(2n)}$ by solving the problem (4)-(6) and compute $\eta = \partial_\nu u^{(2n)} |_{\Gamma_0}$.
3: Find $u^{(2n+1)}$ by solving the problem (7)-(9).
4: If $\frac{||u^{(2m+1)}-u^{(2m)}||_{L^2(\Gamma_0)}}{||u^{(2m+1)}||_{L^2(\Gamma_0)}} < \varepsilon$ then stop.
5: Else, $n \leftarrow n+1$, then
6: Compute $v = v^{(n)} = u^{(2n-1)} |_{\Gamma_0}$ and go to step 2.

The algorithm 1, applied for solving Cauchy problem for Helmholtz equation, have been the subject of several studies (Johansson & Marin, 2009).

In the following, based on the work of Jourhmane & Nachaoui (1996, 1999), we will propose two relaxation alternating algorithms for solving the problem (1)-(3). The aim of these relaxed algorithms is to improve the computational time of the standard algorithm 1, and at the same time maintain the accuracy of the numerical results obtained with this one.

2.2 The relaxed algorithm with fixed parameter factor

The first relaxation algorithm proposed to solve the problem (1) and (2)-(3) has the same computational schemes as the standard alternating algorithm 1 but the Dirichlet condition (5) is relaxed by some relaxation parameter $0 < \theta \leqslant 2$.
This algorithm is summarised as follows,

Algorithm 2:

Step 1: For $m \leftarrow 0$, specify an initial approximation $v = v^0$ of $u |_{\Gamma_0}$, and a relaxation parameter $0 < \theta \leqslant 2$.

Step 2: Find $u^{(2m)}$ by solving the well posed problem

$$\Delta u^{(2m)} - k^2 u^{(2m)} = 0, \quad \text{in } \Omega, \tag{10}$$

$$u^{(2m)} = v^{(m)}, \quad \text{on } \Gamma_0, \tag{11}$$

$$\partial_\nu u^{(2m)} = f_2, \quad \text{on } \Gamma_1, \tag{12}$$

with $v^{(0)} = v^0 |_{\Gamma_0}$ and for $m > 0$,

$$v^{(m)} = \theta u^{(2m-1)} |_{\Gamma_0} \quad +(1-\theta) v^{(m-1)} |_{\Gamma_0} . \tag{13}$$

Step 3: Having construct $u^{(2m)}$ for $x \in \Omega$ and $\eta^{(m)} = \partial_\nu u^{(2m)} |_{\Gamma_0}$ the flux on Γ_0, find $u^{(2m+1)}$ by solving the following well posed problem

$$\Delta u^{(2m+1)} - k^2 u^{(2m+1)} = 0, \quad \text{in } \Omega, \tag{14}$$

$$\partial_\nu u^{(2m+1)} = \eta^{(m)}, \quad \text{on } \Gamma_0, \tag{15}$$

$$u^{(2m+1)} = f_1, \quad \text{on } \Gamma_1. \tag{16}$$

Step 4: If $\frac{||u^{(2m+1)}-u^{(2m)}||_{L^2(\Gamma_0)}}{||u^{(2m+1)}||_{L^2(\Gamma_0)}} < \varepsilon$ then, go to Step 2.

Remark 1. *The value $\theta = 1$ in (13) corresponds to the standard alternating iterative algorithm 1.*

2.3 The relaxed algorithm with dynamic factor

The continuation of the relaxation technique has been successfully progressed by Jourhmane & Nachaoui (1999) for solving Cauchy problem based on finding automatic relaxed factor at each iteration independent of initial guess. This dynamic relaxation factor depends on sequential error between iterations.

This algorithm is summarized as follows,

Algorithm 3:

Step 1: For $m \leftarrow 0$, specify an initial approximation $v = v^0$ of $u\mid_{\Gamma_0}$, and an initial factor θ^0.

Step 2: Find $u^{(2m)}$ by solving the well posed problem

$$\Delta u^{(2m)} - k^2 u^{(2m)} = 0, \quad \text{in } \Omega, \tag{17}$$

$$u^{(2m)} = v^{(m)}, \quad \text{on } \Gamma_0, \tag{18}$$

$$\partial_\nu u^{(2m)} = f_2, \quad \text{on } \Gamma_1, \tag{19}$$

with $v^{(0)} = v^0\mid_{\Gamma_0}$ and for $m > 0$,

$$v^{(m)} = \theta u^{(2m-1)}\mid_{\Gamma_0} + (1-\theta) v^{(m-1)}\mid_{\Gamma_0}. \tag{20}$$

Step 3: Having construct $u^{(2m)}$ for $x \in \Omega$ and $\eta^{(m)} = \partial_\nu u^{(2m)}\mid_{\Gamma_0}$ the flux on Γ_0, find $u^{(2m+1)}$ by solving the following well posed problem

$$\Delta u^{(2m+1)} - k^2 u^{(2m+1)} = 0, \quad \text{in } \Omega, \tag{21}$$

$$\partial_\nu u^{(2m+1)} = \eta^{(m)}, \quad \text{on } \Gamma_0, \tag{22}$$

$$u^{(2m+1)} = f_1, \quad \text{on } \Gamma_1. \tag{23}$$

Step 4: compute $e^{(2m)} = u^{(2m)}\mid_{\Gamma_0} - u^{(2(m-1))}\mid_{\Gamma_0}$ and $e^{(2m+1)} = u^{(2m+1)}\mid_{\Gamma_0} - u^{(2(m-1))}\mid_{\Gamma_0}$

Step 5: compute $\theta^{(m)} = \frac{\langle e^{(2m)}, e^{(2m)} - e^{(2m+1)} \rangle}{||e^{(2m)} - e^{(2m+1)}||_{L^2(\Gamma_0)}}$

Step 6: If $\frac{||u^{(2m+1)} - u^{(2m)}||_{L^2(\Gamma_0)}}{||u^{(2m+1)}||_{L^2(\Gamma_0)}} < \varepsilon$ then stop. Else, $m \leftarrow m+1$, then, go to Step 2.

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3 Numerical results and discussion

In this section, we discuss the numerical results obtained using the all algorithms 1, 2 and 3 proposed for solving a Cauchy problem for Modified Helmholtz equation.

We consider $\Omega =]0,1[\times]0,b[$ where $b > 0$ and we shall solve the following Cauchy problem for Helmholtz equation

$$\Delta u - k^2 u = 0, \quad \text{in} \quad]0,1[\times]0,b[, \tag{24}$$

$$u(x,0) = f_1, \quad \text{on } 0 \le x \le 1, \tag{25}$$

$$\partial_y u(x,0) = f_2, \quad \text{on } 0 \le x \le 1, \tag{26}$$

$$u(0,y) = u(1,y) = 0, \quad \text{on } 0 \le y \le b. \tag{27}$$

We take $\Gamma_1 =]0,1[\times\{0\}$ and $\Gamma_0 =]0,1[\times\{b\}$. For the numerical computations, we particularly choose $b = 0.2, N = 400$ and $M = 80$, and select the boundary data $f_1(x) = u(x, 0)$ on Γ_1 as

$$u(x,0) = \left(3\sin(\pi x) + \frac{\sin(3\pi x)}{19} + 9\exp(-30(x-b)^2\right) x^2(1-x)^2. \tag{28}$$

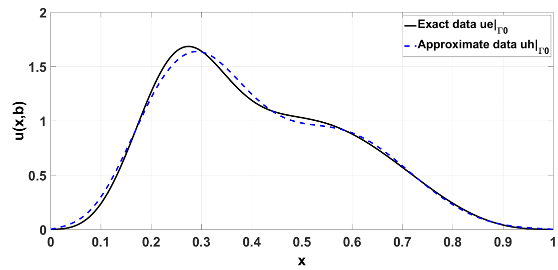
The exact boundary data on Γ_0, used to test the performance of the algorithm, is given by

$$ue(x,b) = 2\left(8\sin(\pi x) + \frac{\sin(3\pi x)}{17} + 20\exp(-50(x-b)^2\right) x^2(1-x)^2. \tag{29}$$

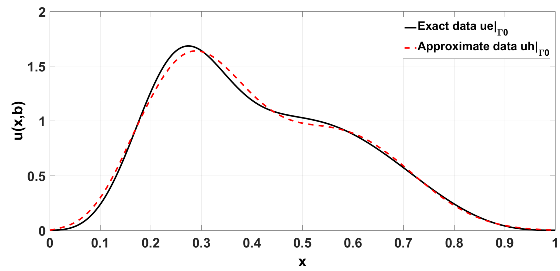
We well solve the above Cauchy problem for Modified Helmholtz equation with two values of k. Namely, $k = \sqrt{15}$, and $k = \sqrt{52}$. The initial guess, $u(x, b) = 0$ on Γ_0 was taken for all algorithms.

For all amgorithms we use the following stopping criteria

$$\frac{\| u^{(2m+1)} - u^{(2m)} \|_{L^2(\Gamma_0)}}{\| u^{(2m+1)} \|_{L^2(\Gamma_0)}} < \varepsilon. \tag{30}$$



(a)



(b)

Figure 1: Reconstructed and exact solutions on $y = b$: (a) algorithm 1, (b), algorithm 2

From fig.2, we see that for the case ($k = \sqrt{15}$), algorithms 1 and 2 produce solutions with the same quality and which are very close to the exact solution .

However, as we can see in fig.2, the relaxed algorithm 2 is faster than algorithm 1 for for a good choice of the parameter θ. In particular the choice of $\theta = \theta_{op} = 1.9$ allows the reduction of the number of iterations by more than half.

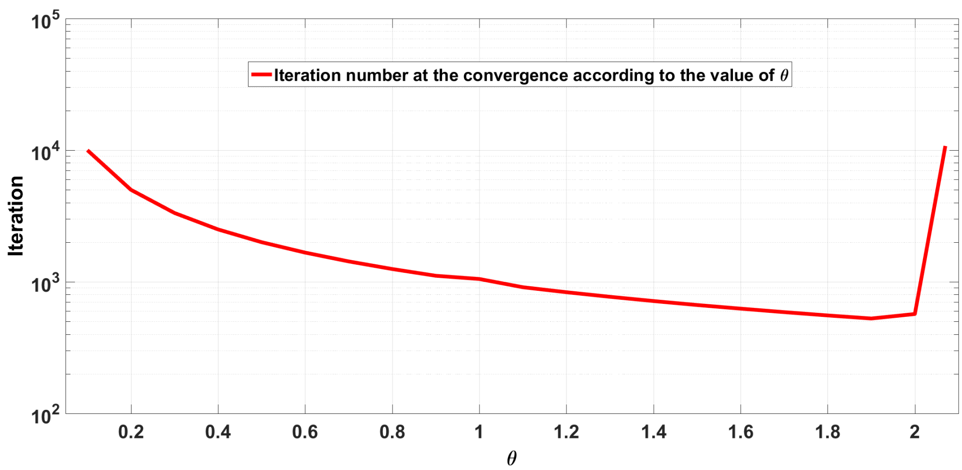


Figure 2: Number of iterations at the convergence for algorithm 2 for $k = \sqrt{15}$.

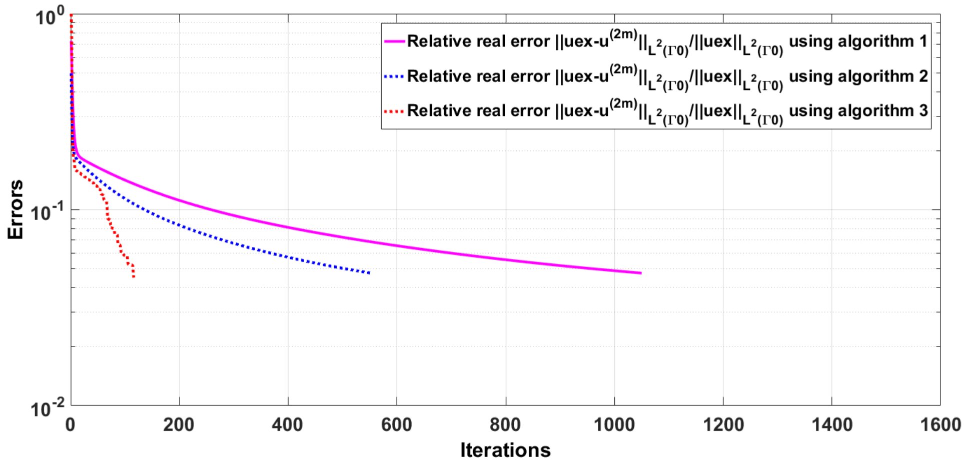


Figure 3: Convergence resultsalgorithms 1, 2 with $\theta = 1.9$ and 3 for $k = \sqrt{15}$.

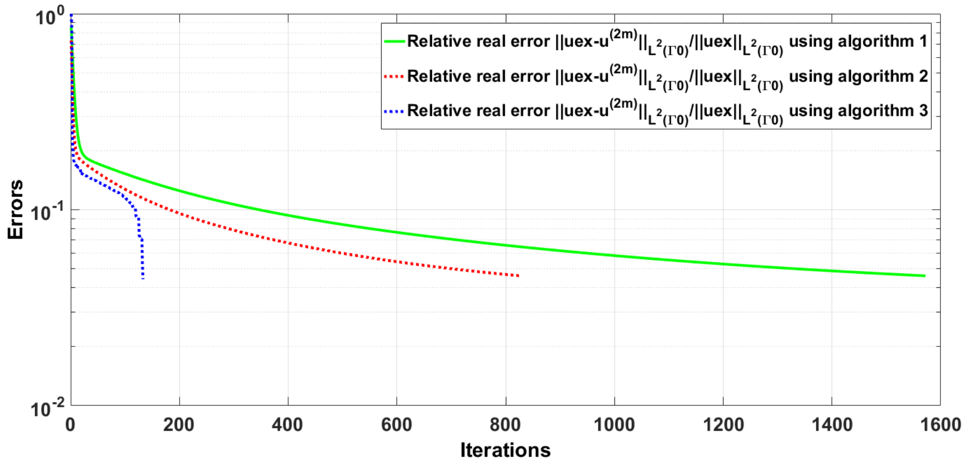


Figure 4: Convergence results for algorithms 1, 2 with $\theta = 1.9$ and 3, for $k = \sqrt{52}$.

To show the efficiency of our algorithms we present other results for the case where the wave number is quite high, $k = \sqrt{52}$ the convergence results are given in fig. 3 and fig. 4. From these figures we see that that, for the same value of the stopping criterion, algorithm 2 and algorithm3 produce solutions that are more precise than that produced by algorithm1. We show

Table 1: Convergence results for $k = \sqrt{15}$ and $\sqrt{52}$.

Wave number	Algorithms	$\|u - ue\|_{L^2(\Omega)}/\|ue\|_{L^2(\Omega)}$	Number of iteration
$k = \sqrt{15}$	Alg 1, $\theta = 1.0$	0.043889	1122
	Alg 2, $\theta_{op} = 1.9$	0.0438868	558
	Alg 3, $\theta_{dynamic}$	0.0439249	**120**
$k = \sqrt{52}$	Alg 1, $\theta = 1.0$	0.0459908	1572
	Alg 2, $\theta_{op} = 1.9$	0.0459895	827
	Alg 3, $\theta_{dynamic}$	0.0442255	**133**

the efficiency of the relaxation algorithms through the table 1. From this table and from the above figures we see that the relaxed algorithm 2 with fixed parameter reduces the convergence rate by half when algorithm3 with dynamic relaxation parameter redices drastically le nombre d'iteration to the convergence (almost 10 times for $k = 15$ and almost 12 times for $k = 52$).

3.1 Numerical stability

In this section, we examine numerically the stability of our approach. We will examine the behavior of the algorithm 2 in the presence of small perturbations in the data. The Dirichlet and Neumann boundary conditions on Γ_1 are perturbed to simulate measurement errors such that

$$f_1^\delta = f_1 + \delta f_1, \qquad f_2^\delta = f_2 + \delta f_2, \tag{31}$$

where $\delta f_1 = f_1 * \delta * (2 * rand - 1)$ and $\delta f_2 = f_2 * \delta * (2 * rand - 1)$ are Gaussian noise with mean zero, generated by an appropriate function "$rand$". While the δ is the noise level.

In order to show that our methods are numerically stable we applied some different level of noise $\delta \in [1 * 10^{-2}, 2 * 10^{-1}]$ to given data and we compare the exact solution ue with the approximate solution without noise and the approximate one u^δ with different noise level δ.

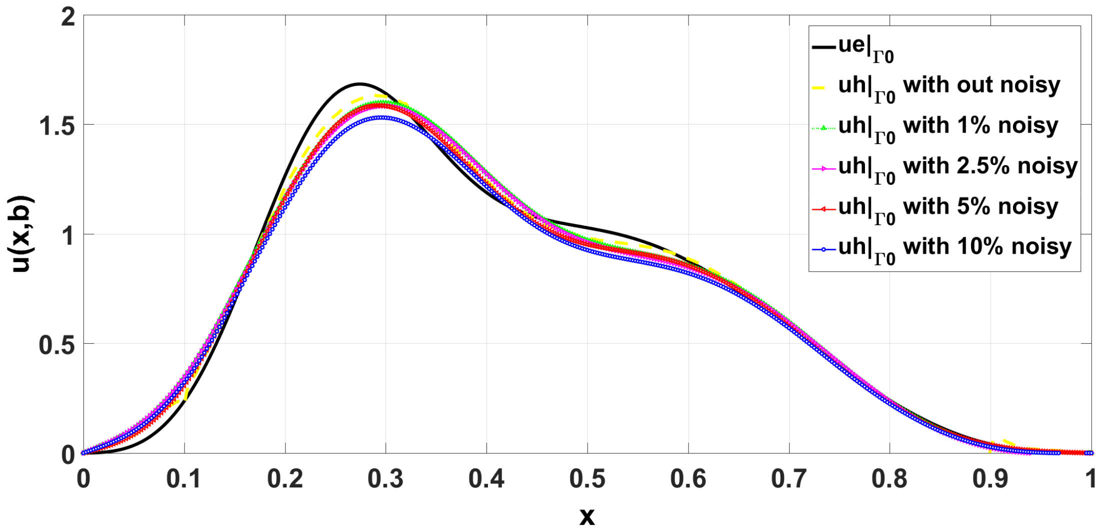


Figure 5: Comparison between the exact solution, the approximate solution without noise and the approximate one with different level of noise using Algorithm 3 with dynamic θ for $k = \sqrt{52}$

In figure 5 we present Comparison between the exact solution, the approximate solution without noise and the approximate one with different level of noise.

As we can see in the figure, the obtained solutions u^δ are close to the exact solution. We can see that the produced errors are of the same order, this implies that the relaxed algorithms are stables.

4 Conclusion

In this paper, we have investigated Cauchy problem associated with Modified Helmholtz equation. We have successfully adapted the relaxed algorithm introduced in Jourhmane & Nachaoui (2002). We presented a variant with a dynamically computed parameter. The results are very convincing the numerical results showed that these algorithms produce more precise solutions than the classical alternative algorithm and that they are very fast. The relaxed alkgorithm 2 with dynamically computed parameter not only rids the user of hazardous choice of relaxation parameter which can compromise the convergence but moreover it drastically reduces convergence. This confirms our prediction that it is an excellent convergence acceleration algorithm.

5 Acknowledgement

The work of the first author, Karzan Berdawood is supported by the split-Site programme at Salahaddin University- Erbil and Nantes University. The work of Fatima Aboud and Mourad

Nachaoui were supported partially by the "Fédération de Recherche Mathématiques des pays de Loire FR-CNRS2962.

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